DIOPHANTINE APPROXIMATION AND DYNAMICAL SYSTEMS 6–8 JANUARY 2018, LA TROBE UNIVERSITY MELBOURNE ABSTRACTS OF TALKS

1. **Dzmitry Badziahin (Sydney, Australia)**

   Title: On potential counterexamles to p-adic Littlewood conjecture.

   Abstract: In 2004 de Mathan and Teulie proposed the problem, called p-adic Littlewood conjecture, which was supposed to be a "simpler" version of the classical Littlewood conjecture. It states as follows: for any prime p and any $x \in \mathbb{R}$, one always has

   $$\liminf_{q\to\infty} q \times |q|_p \times ||qx|| = 0.$$

   I will describe what is known about the set of the (potential) counterexamples x to the p-adic Littlewood conjecture, how small it is and what conditions it must (or must not) satisfy.

2. **Yitwah Cheung (San Francisco, USA)**

   Title: Convergents of simultaneous approximation

   Abstract: A classical theorem of Lagrange characterizes the convergents of the continued fraction of a real number as best approximants of the second kind. In this talk, I will describe a natural generalization of this concept to simultaneous approximation and its relation to the Minkowski Continued Fraction Algorithm. As an application, we show the convergents of any potential counterexample to the Littlewood Conjecture have a lowest terms representation satisfying $\gcd(p_1, q)\gcd(p_2, q) = O(\sqrt{(q)})$.

3. **Sam Chow (York, UK)**

   Title: Title. Bohr sets and multiplicative diophantine approximation

   Abstract: In two dimensions, Gallagher's theorem is a strengthening of the Littlewood conjecture that holds for almost all pairs of real numbers. We prove an inhomogeneous fibre version of Gallagher's theorem, sharpening and making unconditional a result recently obtained conditionally by Beresnevich, Haynes and Velani. The idea is to find large generalised arithmetic progressions within inhomogeneous Bohr sets, extending a construction given by Tao. This precise structure enables us to verify the hypotheses of the Duffin–Schaeffer theorem for the problem at hand, via the geometry of numbers.

4. **Oleg German (Moscow, Russia)**

   Title: Linear forms of a given Diophantine type
   and lattice exponents

   Abstract: The talk is devoted to discussing Diophantine exponents of lattices and connected problems concerning best approximation vectors to linear forms.

   Given a full rank lattice Λ in $\mathbb{R}^d$ we define its Diophantine exponent as

   $$\omega(\Lambda) = \sup\left\{\gamma \in \mathbb{R} \,\middle|\, \Pi(\mathbf{x}) < |\mathbf{x}|^{-\gamma} \text{ admits } \infty \text{ solutions in } \mathbf{x} \in \Lambda\right\},$$

where $\Pi(\mathbf{x}) = |x_1 \cdot \ldots \cdot x_d|^{1/d}$ if $\mathbf{x} = (x_1, \ldots, x_d)$. It follows immediately from Minkowski's convex body theorem that $\omega(\Lambda)$ is nonnegative and it is very natural to conjecture that the spectrum

$$\Omega_d = \left\{ \omega(\Lambda) \,\middle|\, \Lambda \text{ is a lattice in } \mathbb{R}^d \right\}$$

coincides with $[0, \infty]$.

In our talk we shall show how to prove that for each d there is a positive λ such that the ray $[\lambda, +\infty]$ is contained in Ω_d. To this end we shall present a new existence theorem concerning linear forms of a given Diophantine type. Particularly, this theorem implies that, given a positive ω, there is a linear form $L(\,\cdot\,)$ in $\mathbb{R}^d$ such that for each $\mathbf{x}$ that is a best approximation to $L(\,\cdot\,)$ we have

$$|L(\mathbf{x})| \cdot |\mathbf{x}|^{d-1} \asymp |\mathbf{x}|^{-\omega},$$

while for each $\mathbf{x}$ that is not a best approximation vector to $L(\,\cdot\,)$ we have

$$|L(\mathbf{x})| \cdot |\mathbf{x}|^{d-1} \gg |\mathbf{x}|^{-\omega + c(\omega)}$$

with some positive $c(\omega)$.

5. **Andy Hone (Canterbury, UK)**

Title: Curious continued fractions

Abstract: There are relatively few transcendental numbers for which the continued fraction expansion is explicitly known. Here we present two new families of continued fractions for Engel series - sums of reciprocals - which arise from integer sequences generated by nonlinear recurrences with the Laurent property (one of the key features of Fomin and Zelevinsky's cluster algebras). Using the double exponential growth of the sequences, we show that the sum of such Engel series is transcendental. If time allows, we will make some remarks about continued fraction expansions in hyperelliptic function fields, and some related discrete integrable systems.

6. **Oleg Karpenkov (Liverpool, UK)**

Title: Gauss-Kuzmin statistics for faces of Klein polyhedra

Abstract: In this talk we discuss a multidimensional generalization of Gauss-Kuzmin statistics. Currently, an ergodic approach to the question has not been developed. In particular this is due to the following observation: to find an appropriate generalization of the Gauss map for the multidimensional continued fractions is a hard problem. In fact, this problem can be avoided by considering Gaus-Kuzmin statistic from the Moebius measure perspectives. Our goal is to show this in our talk.

7. **Simon Kristensen (Aarhus, Denmark)**

Title: Applications of uniform distribution to Diophantine problems

Abstract: The theory of uniform distribution modulo 1 is neatly placed in the interface between number theory and dynamical systems. I will give some problems and partial results, showing how the theory of uniform distribution can be applied to Diophantine questions, including problems in Diophantine approximation, problems in the theory of normal numbers and to questions of irrationality of values of series expansions of real numbers.

8. **Bing Li (Guangzhou, China)**

   Title: Simultaneous shrinking target problems for $\times 2$ and $\times 3$

   Abstract: We consider the simultaneous shrinking target problems for $\times 2$ and $\times 3$. We obtain the Haisdorff dimensions of the intersection of two well approximable sets and also of the set of points whose orbits approach a given point simultaneously for these two dynamical systems. It is a joint work with Lingmin Liao.

9. **Seonhee Lim (Seoul, Korea)**

   Title: Hausdorff dimension bound for badly approximable vectors and grids

   Abstract: :We show that for any epsilon>0, the Hausdorff dimension of epsilon-badly approximable vectors in a inhomogeneous Diophantine approximation is not full and has a bound depending on epsilon. We use a relative version of the uniqueness of the entropy-maximizing measure in an appropriate homogeneous space. This is a joint work with Uri Shapira and Nicolas de Saxce

10. **Tanja Schindler (Canberra, Australia)**

    Title: Limit theorems on counting large continued fraction digits.

    Abstract: Inspired by a result of Galambos on Lüroth expansions we give a refinement of the famous Borel-Bernstein Theorem for continued fractions and - closely related to this - a Central Limit Theorem for counting large continued fraction digits. Some of the results which are true for the continued fraction digits can be transferred to diophantine approximants. This is joint work with Marc Kesseböhmer.

11. **Johannes Schleischitz (Ottawa, Canada)**

    Title: Uniform approximation to a certain class of manifolds

    Abstract: For $\underline{\zeta} = (\zeta_1, \dots, \zeta_n)$ a real vector, define $\widehat{\lambda}(\underline{\zeta})$ and $\widehat{w}(\underline{\zeta})$ resp. as the supremum of real numbers λ and w such that

$$1 \leq q \leq Q, \qquad \max_{1 \leq i \leq n} |q\zeta_i - p_i| \leq Q^{-\lambda}$$

and

$$\max_{0 \leq i \leq n} |p_i| \leq Q, \qquad 0 < |p_0 + p_1\zeta_1 + \dots + p_n\zeta_n| \leq Q^{-w}$$

resp. have a solution $(p_1, \dots, p_n, q) \in \mathbb{Z}^{n+1}$ for all large Q. By Dirichlet's Theorem and the theory of continued fractions, we have $\frac{1}{n} \leq \widehat{\lambda}(\underline{\zeta}) \leq 1$ and $n \leq \widehat{w}(\underline{\zeta}) \leq \infty$, and these bounds are sharp. However, the situation changes drastically when restricting to manifolds. For C some curve in $\mathbb{R}^n$ let

$$\widehat{\lambda}(C) = \sup_{\underline{\zeta} \in C} \widehat{\lambda}(\underline{\zeta}), \qquad \widehat{w}(C) = \sup_{\underline{\zeta} \in C} \widehat{w}(\underline{\zeta}).$$

Studying approximation to real numbers by algebraic integers, Davenport and W. Schmidt in 1969 treated the case of successive powers $\zeta_j = \zeta^j$, i.e. $C = C_n$ is the Veronese curve in dimension n. They showed the upper bounds $\widehat{\lambda}(C_n) \leq \frac{2}{n}$ and $\widehat{w}(C_n) \leq 2n - 1$. Only in the case $n = 2$ the values $\widehat{\lambda}(C_n), \widehat{w}(C_n)$ are known due to Roy. The talk aims to both improve and generalize the bounds for larger n. The best known upper bounds are still only slightly better than those from Davenport and Schmidt. Concerning simultaneous approximation to the Veronese curve, at the moment (unconditionally) we know $\widehat{\lambda}(C_n) \leq \frac{2}{n+1}$ for odd n (M. Laurent), of order $\widehat{\lambda}(C_n) \leq \frac{2}{n} - \frac{2}{n^2} + O(n^{-3})$ for even n, and for the dual problem $\widehat{w}(C_n) \leq 2n - 2 + O(n^{-1})$. Further improvements of these bounds possibly lead to a better understanding of approximation to real numbers by algebraic numbers/integers, connected to Wirsing's open problem if any transcendental real number can be approximated of order $n+1$ by algebraic numbers of degree at most n. It is for example known that the order of approximation is bounded from below by $\widehat{\lambda}(C_n)^{-1}+1$.

We will also study the more general case of uniform approximation to curves parametrized by polynomials with rational coefficients, i.e. given by

$$C = \{(P_1(T), \dots, P_n(T)) \in \mathbb{R}^n : T \in \mathbb{R}\}, \qquad P_i \in \mathbb{Q}[T]. \tag{1}$$

The special case of P_i of degree at most two is understood thanks to Roy, and bounds for the case $= 2$ and $P_1(T) = T^k, P_2(T) = T^l$ concerning approximation to vectors of the form (ζ^k, ζ^l) for k, l integers have recently been given by Batzaya. Non-trivial upper bounds for general curves as in (1), and key ideas of their proofs, will be presented. This is work in progress. The question if our bounds are close to the optimal value is very open. The talk will also address earlier (partly metric) results on best approximation to curves in (1), the involved methods are similar.

12. **David Simmons (York, UK)**

Title: A variational principle in the parametric geometry of numbers, with applications to metric Diophantine approximation

Abstract: The parametric geometry of numbers, introduced by Schmidt and Summerer, is a framework for analyzing the Diophantine properties of a vector in terms of the successive minima of a certain one-parameter family of convex regions (or equivalently of a certain family of lattices) defined in terms of that vector. We generalize this framework to the setting of matrix approximation, and we calculate the Hausdorff and packing dimensions of certain sets defined in terms of the parametric geometry of numbers. One of the many applications of our theorem is a proof of the conjecture of Kadyrov, Kleinbock, Lindenstrauss, and Margulis stating that the Hausdorff dimension of the set of singular $m \times n$ matrices is equal to $mn(1 - \frac{1}{m+n-1})$. This work is joint with Tushar Das, Lior Fishman, and Mariusz Urbański.

13. **Michel Waldschmidt (Paris, France)**

Title: Two applications of Diophantine approximation to dynamical systems.

Abstract: The behavior of a holomorphic dynamical system near a fixed point depends on a Diophantine condition arising in the works of Liouville, Thue, Siegel and Roth on the rational approximation to algebraic numbers. Schmidt Subspace Theorem is a far reaching generalization of the Thue–Siegel–Roth Theorem; one of its many consequences is a result on the iterates of an endomorphism of a vector space.

14. **Bao-Wei Wang (Wuhan, China)**

Title: Diophantine approximation, continued fractions and Hausdorff dimension

Abstract: Continued fraction expansion is a vital tool in studying the problems in Diophantine approximation. We will first talk about the relationship between Diophantine approximation and continued fraction expansion; then we relate the three fundamental theories in metric Diophantine approximation (Dirichlet's theorem, Khintchine's theorem and Jarnik's theorem) to the questions in continued fractions. These enable us to improve the classic results by using continued fractions.